Topology Vol. 16, pp. 199–201. Pergamon Press, 1977. Printed in Great Britain

HARMONIC MAPPINGS OF UNIFORM BOUNDED DILATATION

LUC LEMAIRE†

(*Received* 13 *September* 1976)

LET M AND M' be compact Riemannian manifolds and suppose that the sectional curvature of M' is negative. We show that for all $K \in R$, there is only a finite number of non-constant harmonic mappings from M to M' of dilatation bounded by K. When M and M' are in addition almost Kaehlerian, this implies that the number of non-constant almost complex maps from M to M' is finite.

Definitions

Let M, g and M', g' be compact connected C^∞ Riemannian manifolds of dimensions n and n' and f: $M \to M'$ a C^∞ mapping.

Definition 1[1]. $f \in C^\infty(M, M')$ is harmonic iff it is a critical point of the energy functional

$$E(f) = \frac{1}{2}\int |df|^2 v_g$$

where $df(m)$: $T_mM \to T_{f(m)}M'$ is the differential of f at m and v_g denotes the volume element associated to g.

We shall use a definition of mapping of bounded dilatation which is slightly more general than the one introduced in [2]. For each m in M, the pull-back f^*g' of the metric on $T_{f(m)}M'$ is a symmetric semidefinite quadratic form on T_mM. Let $k \leq n, n'$ be its rank. We can choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_mM such that $f^*g' = \sum_{i=1}^{k} \lambda_i\omega_i \otimes \omega_i$, where ω_i is the dual 1-form of e_i and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$.

Definition 2[2]. $l_1 = (\lambda_1/\lambda_2)^{1/2}$ is the (first) dilatation of f at m.

l_1 is by definition greater or equal to 1.

Definition 3. We say that the dilatation of f is bounded by K iff at each point of M we have $l_1 \leq K$ or $df = 0$.

Remark 1. We admit in this definition that the rank varies from a point to another, and that it takes the value 0. But it can never take the value 1, since l_1 would then be infinite.

Remark 2. If M and M' are surfaces, a mapping of dilatation bounded by K is a K-quasiconformal mapping.

Finiteness Theorem

THEOREM 1. *Let M and M' be compact Riemannian manifolds and suppose the sectional curvature of M' strictly negative. Let $K \geq 1$. Then there is only a finite number of non-constant harmonic mappings from M to M' of dilatation bounded by K.*

Proof. By the assertion (I) of [3], there can be only one non-constant harmonic map of bounded dilatation in a homotopy class of maps from M to M'. This follows from the hypothesis on the curvature of M' and the fact that the image of such a map cannot be a geodesic since its rank cannot be 1. So we just have to prove that only a finite number of homotopy classes can contain a non-constant harmonic map of dilatation bounded by K.

M and M' being compact, we can find positive numbers A and B such that $-A$ is a lower bound for the sectional curvature of M and $-B$ an upper bound for that of M'. One can then check that the proof of theorem 4.1 of [2] applies to the definition used here (definition 3) and that

†Aspirant au Fonds National Belge de la Recherche Scientifique.

the condition on the dilatation implies

$$|df|^2 \leq \frac{n-1}{2} N^2 K^2 \frac{A}{B} \equiv C^2$$

where $N = \min(n, n')$. So the map f can at most multiply distances by C. The principle of the proof of this result goes back to [5, theorem 6], where it is applied to the case of quasiconformal maps between surfaces. We use it to study the homotopy classes in the following way.

Since the sectional curvature of M' is negative, the Cartan–Hadamard theorem asserts that the homotopy groups $\Pi_i(M')$ are trivial for $i \geq 2$. By theorem 8.1.11 of [8], this implies that the homotopy classes of maps f from M to M' are parametrized by the conjugacy classes of the induced homomorphisms $f_*: \Pi_1(M) \to \Pi_1(M')$.

Call U and U' the universal coverings of M and M' and choose a point P of U and a fundamental domain $\mathcal{F}'$ of U'. Every map $f: M \to M'$ can be lifted to a map $\bar{f}: U \to U'$ such that $\bar{f}(P) \in \mathcal{F}'$. An element γ of $\Pi_1(M)$ can be seen as an automorphism of the covering U and for each γ the map $\bar{f}$ verifies the relation $\bar{f} \circ \gamma = \bar{f}_*(\gamma) \circ \bar{f}$, where $\bar{f}_*$ is one of the conjugates of f_*, depending on the choice of $\mathcal{F}'$.

Let $S = \{\rho_0 = 1, \rho_1, \ldots, \rho_s\}$ be a set of generators of $\Pi_1(M)$ and put $P_r = \rho_r(P)$, $r = 0, \ldots, s$. The images of the P_r's by a map $\bar{f}$ are contained in $\{\alpha \cdot \bar{f}(P) | \alpha \in \Pi_1(M')\}$.

The set S being finite, we can find a bounded connected domain D of U containing all P_r's. Since the maps that we consider send P in $\mathcal{F}'$ and multiply the distances by at most a fixed constant C, the images of D by these maps are contained in a bounded set D' of U'. Since D' is bounded, the set

$$T = U_{\bar{f}}\{\alpha \in \Pi_1(M') | \alpha \cdot \bar{f}(P) \in D'\}$$

is finite.

The conjugacy classes of the associated homomorphisms f_* are characterized by the restrictions $f_{*|S}: S \to T$. Since S and T are finite, the number of these classes is also finite, and so is the number of homotopy classes containing a non-constant harmonic map of dilatation bounded by K.

Remark 3. In view of Satz 5.9 of [4], one might ask whether theorem 1 can be extended to the case where M' is a product of manifolds of negative sectional curvature. We shall show by an example that it is not the case.

Let N and N' be compact Riemannian manifolds with N' of negative sectional curvature and such that there is an infinity of homotopy classes of maps from N to N'. For instance, N and N' could be Riemann surfaces with genus $N \geq$ genus $N' \geq 2$. (Indeed, a map between these surfaces can twist a handle of N around a handle of N' an arbitrary number of times.)

By [1], every homotopy class of maps from N to N' contains a harmonic map and we can choose an infinite sequence $h^{(t)}$ of distinct non-constant harmonic maps. (By theorem 1, the associated ratios $\lambda_1^{(t)}/\lambda_2^{(t)}$ form an unbounded set of numbers). The set of maps $h^{(t)} \times h^{(t)}: N \times N \to N' \times N'$ is then an infinite set of distinct non-constant harmonic maps, and their dilatation is always 1 since the eigenvalues of $(h^{(t)} \times h^{(t)})^*(g' \times g')$ are $(\lambda_1^{(t)}, \lambda_1^{(t)}, \ldots, \lambda_k^{(t)}, \lambda_k^{(t)})$.

Almost complex maps

Proposition 1. *Let M, J, g and M', J', g' be almost Hermitian manifolds and f an almost complex (or almost holomorphic) map from M to M'. For each $m \in M$, there exists an orthonormal basis $\{e_i\}$ of T_mM such that $e_{2j} = Je_{2j-1}$ and $f^*g' = \sum_{i=1}^{k} \lambda_i \omega_i \otimes \omega_i$, with $\lambda_{2j} = \lambda_{2j-1}$ and $\lambda_1 = \lambda_2 \geq \lambda_3 = \ldots \geq \lambda_{k-1} = \lambda_k > 0$.*

Proof. Consider an orthonormal basis $\{e_i\}$ of T_mM such that $f^*g' = \sum_{i=1}^{k} \lambda_i \omega_i \otimes \omega_i$, with $\lambda_i \geq \lambda_{i+1}$. Since f is almost complex and M' almost Hermitian, we have for $X, Y \in T_mM$:

$$\begin{aligned} f^*g'(JX, JY) &= g'(df \cdot JX, df \cdot JY) \\ &= g'(J' df \cdot X, J' df \cdot Y) \\ &= g'(df \cdot X, df \cdot Y) \\ &= f^*g'(X, Y). \end{aligned}$$

So in particular, $f^*g'(Je_1, Je_1) = \lambda_1$. Since Je_1 is normal to e_1, it is a combination of the e_i's, $i \geq 2$. Since $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, λ_2 must be equal to λ_1.

If $\lambda_3 \neq \lambda_2$, then $e_2 = Je_1$ and we can consider the space generated by $e_3, \ldots, e_n$ and apply the same reasoning to prove that $\lambda_3 = \lambda_4$.

If $\lambda_2 = \lambda_3 = \cdots = \lambda_p$, then Je_1 is in the space generated by $e_2, \ldots, e_p$. By a rotation of that space (which preserves the restriction of f^*g', equal to the restriction of $\lambda_2 g$) we can replace e_2 by Je_1. We then proceed as above to prove that $\lambda_3 = \lambda_4$.

The proposition follows from a repetition of this argument.

Let F denote the fundamental 2-form of the almost Hermitian manifold M. Recall [7; IV, 16, c] that M is called special if $dF^{(n/2)-1} = 0$ and special of pure type if $(dF)_{1,2} = 0$. An almost Kaehlerian manifold is always special of pure type since its fundamental 2-form is closed.

Proposition 2. *If M is a compact special almost Hermitian manifold and M′ a compact special almost Hermitian manifold of pure type and of negative sectional curvature, then there is only a finite number of non-constant almost complex maps from M to M′.*

Indeed, by prop. IV, 16, d of [7], an almost complex map is harmonic and by prop. 1, its first dilatation is 1.

Remark 4. As we have seen, a special case of proposition 2 is obtained by supposing M and M' almost Kaehlerian and M' of negative curvature.

In the case of almost complex maps, we can consider a product of manifolds of negative curvature and obtain the following analogue of [4, Satz 5.9 (2)].

Corollary 1. *Let M be a compact special almost Hermitian manifold and M′ a product of compact special almost Hermitian manifolds of pure type and of negative sectional curvature. Then there is only a finite number of almost complex maps from M to M′ whose projections on all factors are non-constant.*

Indeed, the maps followed by the projections satisfy the hypothesis of proposition 2. Numerous finiteness results for holomorphic maps can be found in [6, §8].

I wish to express my thanks to James Eells for his help during this research.

REFERENCES

1. J. Eells and J. Sampson: Harmonic mappings of Riemannian manifolds, *Am. J. Math.* **86** (1964), 109–160.
2. S. I. Goldberg, T. Ishihara and N. C. Petridis: Mappings of bounded dilatation of Riemannian manifolds, *J. diff. Geom.* **10** (1975), 619–630.
3. P. Hartman: On homotopic harmonic maps, *Can. J. Math.* **19** (1967), 673–687.
4. W. Kaub: Hyperbolische komplexe Räume, *Annls. Inst. Fourier Univ. Grenoble* **18**, fasc. 2 (1968), 303–330.
5. P. Kiernan: Quasiconformal mappings and Schwarz's lemma, *Trans. Am. Math. Soc.* **148** (1970), 185–197.
6. S. Kobayashi: Intrinsic distances, measures and geometric function theory, *Bull. Am. Math. Soc.* **82** (1976), 357–416.
7. A. Lichnerowicz: Applications harmoniques et variétés Kählériennes, *Symp. Math.* **3** (1970), 341–402.
8. E. Spanier: *Algebraic topology*, McGraw-Hill (1966).

University of Warwick.